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THEORY OF AIRCRAFT STRUCTURAL MODELS SUBJECT TO

AERODYNAMIC HEATING AND EXTERNAL LOADS

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SUMMARY

The problem of investigating the simultaneous effects of transient aerodynamic heating and external loads on aircraft structures for the purpose of determining the ability of the structure to withstand flight to supersonic speeds is studied. By dimensional analyses it is shown that:

(a) A structural model geometrically similar to the aircraft and constructed of the same materials as the aircraft will be thermally similar to the aircraft with respect to the flow of heat through the structure

(b) The thermal stresses and deformations of the structural model will be similar to those of the aircraft when the structural model is constructed at the same temperature as the aircraft.

(c) The stresses and deformations of the structural model due to external loads will be similar to those of the aircraft.

(d) By aerodynamic means the structural model can automatically be subjected to heating and cooling that correctly simulate the aerodynamic heating of the aircraft, except with respect to angular velocities and angular accelerations, without requiring determination of the heat flux at each point on the surface and its variation with time.

(e) The similitude of aerodynamic forces, moments, and pressures acting on the aerodynamically heated structural model to those acting on the aircraft is determined for the case of zero angular velocity and zero angular acceleration, so that the structural model may be subjected to the external loads required for simultaneous simulation of stresses and deformations due to external loads.

INTRODUCTION

The phenomenon of aerodynamic heating of supersonic airplanes and guided missiles imposes the new design requirement that every component of the aircraft be capable of withstanding the adverse effects of the elevated temperatures and rates of change of temperature to which it is subjected in flight. Consideration is restricted herein to this problem with respect to the aircraft structure.

Structural materials exhibit markedly different variations of strength with increase in temperature. Below the temperature of complete strength loss, the aircraft structure may experience serious decrease in strength and suffer other detrimental effects by a number of processes. If the structure were to have a uniform high temperature throughout but be composed of materials of different coefficients of thermal expansion, stresses caused by unequal expansion would be generated in the structure; however, even if all parts had the same coefficient of thermal expansion, the rapid changes of velocity, altitude, and attitude of supersonic aircraft would produce transient aerodynamic heating which, together with finite thermal diffusivity (even of metallic structures), would result in unequal temperatures that would produce stresses as a result of unequal thermal expansion. The loss of material strength at elevated temperatures and the development of thermal stresses are attended by thermal distortion and altered elastic properties of the structure. These changes give rise to deflections that alter the externally applied aerodynamic loads as well as the distribution of load between the component parts of the structure. Dependent upon the type of structure, various kinds of thermally induced aeroelastic problems such as panel flutter may result.

If reliable computations of the transfer of heat between the air and every element of external surface of the aircraft at every instant of time during a flight and, in addition, computations of the flow of heat through the complex aircraft structure were possible, computations of the thermal stresses in the structure, the remaining strength, the thermal distortion, the alteration of aerodynamic characteristics and loads by the thermal distortion, the modification of the aeroelastic characteristics, and other important effects of transient aerodynamic heating might then be possible. Because of the enormousness of such a task, it appears that some other method of approach may be more practical.

It is theoretically conceivable that by means of a dimensional analysis the necessary scaling parameters might be developed whereby a structural model of the actual aircraft could be built and subjected to aerodynamic-heating tests with attendant benefits analogous to those obtained with models in both free-flight and wind-tunnel aerodynamic investigations. Appropriate measurements upon such a structural model would yield the information sought, the physical phenomena involved being

thereby compelled in a sense to compute the wanted information. The present paper constitutes a first endeavor to develop rigorously by dimensional analysis the theory and attendant scaling laws for the construction and testing of such aerodynamic-heating structural models having the requisite similarity to any given aircraft flying any prescribed path.

For logical development of the theory, it has been found desirable to subdivide the presentation into the following parts:

I. Thermal Similitude - This part includes the development of the equations defining the conditions under which there exists similitude of temperature, temperature distribution, heat content, and flow of heat in the model structure to that in the aircraft structure.

II. Thermal Stress and Deformation Similitude - This part presents the development of the equations expressing the conditions under which there exists similitude of thermally induced stresses and deformations between the model and aircraft structures while thermal similitude is preserved.

III. External Load Stress and Deformation Similitude - This part presents the development of the equations expressing the conditions under which there exists similitude of stresses and deformations due to external loads between the model and aircraft structures while thermal similitude and similitude of thermally induced stresses and deformations are preserved.

IV. Heating the Structural Model - This part presents development of the equations expressing the conditions under which the structural model can be subjected, by aerodynamic means, to the requisite heating and cooling in simulation of the aerodynamic heating experienced by the aircraft in flight, without requiring determination of the heat transfer to each point on the surface of the aircraft and model.

V. External Loading of Structural Model When Aerodynamically Heated - This part presents the development of the equations relating the aerodynamic forces, moments, and pressures imposed on the structural model when aerodynamically heated to those of the aircraft to permit adjustment of the loads to loads required for similitude of stresses and deformations due to external loads.

SYMBOLS

A   coefficient of thermal expansion of structure material,
                      $1/^{\circ}R$

a_{∞} free-stream sound velocity, ft/sec

$a_1, a_2, \ldots$ exponents

$b_1, b_2, \ldots$ exponents

C   specific heat of structure material,
                      $(Btu)(ft)/(lb)(sec^2)(^{\circ}R)$

c_p free-stream specific heat at constant pressure,
                      $ft^2/(sec^2)(^{\circ}R)$

$c_1, c_2, \ldots$ exponents

D   mass density of structure material, $lb\text{-}sec^2/ft^4$

$d_1, d_2, \ldots$ exponents

E   modulus of elasticity in tension or compression, lb/ft^2

$e_1, e_2, \ldots$ exponents

F   external concentrated load, lb

G   modulus of elasticity in shear, lb/ft^2

H   heat, Btu

·h   aerodynamic heat-transfer coefficient, $Btu/(ft^2)(sec)(^{\circ}R)$

J   aerodynamic moment, lb-ft

K   thermal conductivity of structure material,
                      $Btu/(ft)(sec)(^{\circ}R)$

k   free-stream thermal conductivity, $lb/(sec)(^{\circ}R)$

L   dimensional concept length

l linear dimension indicative of size, ft

M	dimensional concept mass
N	external moment, ft-lb
n	scale factor
P	external distributed load, lb/ft^2
p	aerodynamic pressure, lb/ft^2
p_∞	free-stream pressure, lb/ft^2
Q	dimensional concept heat
q	heat flux, lb/ft-sec
R	aerodynamic force, lb
r	arbitrary exponent
T	dimensional concept time
t	time, sec
V_∞	free-stream velocity, ft/sec
α	angular orientation, radians
$\dot{\alpha}$	angular velocity, radians/sec
$\ddot{\alpha}$	angular acceleration, radians/sec^2
γ	ratio of free-stream specific heats
$\Delta\theta$	temperature change from assembly temperature, ^{O}R
δ_θ	thermal linear deformation, ft
δ_L	load linear deformation, ft
Θ	dimensional concept temperature
θ	temperature, ^{O}R
κ	thermal diffusivity of structure material, ft^2/sec
μ_∞	free-stream absolute viscosity, lb-sec/ft^2

ν Poisson's ratio

ρ_∞ free-stream mass density of gas, $lb\text{-}sec^2/ft^4$

σ_L load tensile or compressive stress, lb/ft^2

σ_p proportional limit in tension or compression, lb/ft^2

σ_θ thermal tensile or compressive stress, lb/ft^2

τ_L load shear stress, lb/ft^2

τ_p proportional limit in shear, lb/ft^2

τ_θ thermal shear stress, lb/ft^2

Subscripts:

a   aircraft

m   model

I. THERMAL SIMILITUDE

Formulation of Problem

The objective of this section is to investigate what thermal similarity can exist between an aircraft structure and a model thereof under transient heating or cooling. By thermal similarity is meant the relations of the temperature, temperature distribution, heat content, and flow of heat in the model to those in the aircraft structure.

Physical Quantities Involved

The assumption is made that the flow of heat inside the structure is predominantly by the process of thermal conduction through the solid material so that transfer of heat by radiation from one part to another and convective transfer of heat through the air inside the structure from one part to another are negligible. Accordingly, the transient flow of heat is assumed to be dependent upon the following physical quantities having the indicated consistent engineering units and dimensions in terms of length L, mass M, time T, temperature Θ, and heat Q. The consideration of temperature Θ and heat Q as distinct dimensions is justified in that in the phenomena here considered they retain their identity.

Symbol	Quantity	Units	Dimensions
l	Linear dimension indicative of size	ft	L
θ	Temperature	°R	θ
D	Mass density	lb-sec^2/ft^4	$L^{-3}M$
C	Specific heat of material	(Btu)(ft)/(lb)(sec^2)(°R)	$M^{-1}\theta^{-1}Q$
K	Thermal conductivity	Btu/(ft)(sec)(°R)	$L^{-1}T^{-1}\theta^{-1}Q$
H	Heat	Btu	Q
t	Time	sec	T

Dimensional Analysis

By dimensional analysis (refs. 1 and 2), the conditions of transient thermal similitude are found as follows. Selecting length l, time t, temperature θ, mass density D, and heat H as the primary physical quantities, the resulting π-factors are

$$C \, l^{a_1} \, t^{b_1} \, \theta^{c_1} \, D^{d_1} \, H^{e_1}$$

$$K \, l^{a_2} \, t^{b_2} \, \theta^{c_2} \, D^{d_2} \, H^{e_2}$$

The exponents a_1, a_2, b_1, b_2, . . . are to be determined by the conditional equations upon L, M, T, θ, and Q.

The conditional equations of the first π-factor containing C are obtained by substituting the dimensions of the physical quantities giving

$$\left(M^{-1}\theta^{-1}Q\right)(L)^{a_1}(T)^{b_1}(\theta)^{c_1}\left(L^{-3}M\right)^{d_1}(Q)^{e_1}$$

The conditional equations are then:

$$L: \qquad a_1 - 3d_1 = 0$$

$$M: \qquad -1 + d_1 = 0$$

$$T: \quad b_1 = 0$$

$$\Theta: \quad -1 + c_1 = 0$$

$$Q: \quad 1 + e_1 = 0$$

from whence

$$a_1 = 3$$

$$b_1 = 0$$

$$c_1 = 1$$

$$d_1 = 1$$

$$e_1 = -1$$

The dimensionless parameter resulting from the first π-factor is then $Cl^3\Theta D/H$.

Similarly, substitution of the dimensions into the second π-factor containing K gives

$$\left(L^{-1}T^{-1}\Theta^{-1}Q\right)(L)^{a_2}(T)^{b_2}(\Theta)^{c_2}\left(L^{-3}M\right)^{d_2}(Q)^{e_2}$$

The conditional equations are then:

$$L: \quad -1 + a_2 - 3d_2 = 0$$

$$M: \quad d_2 = 0$$

$$T: \quad -1 + b_2 = 0$$

$$\Theta: \quad -1 + c_2 = 0$$

$$Q: \quad 1 + e_2 = 0$$

from whence

$$a_2 = 1$$

$$b_2 = 1$$

$$c_2 = 1$$

$$d_2 = 0$$

$$e_2 = -1$$

so that the dimensionless parameter resulting from the second π-factor is $Klt\theta/H$.

The solution yielded by the dimensional analysis is then

$$f\left(\frac{Cl^3\theta D}{H}, \quad \frac{Klt\theta}{H}\right) = 0 \qquad (1)$$

Similitude Equations

For purpose of identification, the aircraft structural quantity is denoted with the subscript a and the model structural quantity with the subscript m; the similitude equations relating the transient heating of the aircraft and model structures may then immediately be written from equation (1) as

$$\frac{C_m l_m^3 \theta_m D_m}{H_m} = \frac{C_a l_a^3 \theta_a D_a}{H_a} \qquad (2)$$

$$\frac{K_m l_m t_m \theta_m}{H_m} = \frac{K_a l_a t_a \theta_a}{H_a} \qquad (3)$$

Equations (2) and (3) must be simultaneously satisfied. The conditions under which this requirement is possible may be examined as follows. Substituting equation (2) into equation (3) yields

$$\frac{K_m t_m}{C_m D_m l_m^2} = \frac{K_a t_a}{C_a D_a l_a^2} \qquad (4)$$

Consider that, for every point in the aircraft structure, there must be a corresponding point in the model structure; then, a scale n between the aircraft and model structures may be defined as

$$l_m = n l_a \tag{5}$$

Substituting equation (5) into equation (4) gives

$$\frac{K_m}{C_m D_m} t_m = n^2 \frac{K_a}{C_a D_a} t_a \tag{6}$$

In equation (6) it is observed that transient thermal similitude is dependent upon the ratio of thermal properties of the materials represented by K/CD. This ratio is recognized·as being the thermal diffusivity κ so that equation (6) may be written as

$$\kappa_m t_m = n^2 \kappa_a t_a \tag{7}$$

In the analytical theory of the transient conduction of heat in solids (refs. 3 and 4), it is shown that the phenomenon is dependent only upon the one physical property of the material of thermal diffusivity. Thus, this result indicated by equations (6) and (7) might have been foreseen. The thermal diffusivity κ, or its components, particularly the thermal conductivity K and the specific heat C, have received little investigation especially for structural materials. This is probably due in large measure to the fact that such investigations as have been made, primarily in the fields of calorimetry and metallurgy, have revealed that most materials exhibit significant anomalies in these thermal properties as functions of temperature. Consequently, it appears impractical to fulfill equation (7) over any appreciable range of temperature other than by constructing the model of identically the same material as used in the aircraft. Assume, then, that corresponding parts of the model and aircraft are constructed of identical materials, so that

$$K_m = K_a \tag{8}$$

$$C_m = C_a \tag{9}$$

$$D_m = D_a \tag{10}$$

or

$$\kappa_m = \kappa_a \tag{11}$$

which implies, since these thermal properties are assumed to be functions of temperature, that

$$\theta_m = \theta_a \tag{12}$$

The substitution of equations (8), (9), and (10) into equation (6); or the substitution of equation (11) into equation (7) then gives

$$t_m = n^2 t_a \tag{13}$$

This means that an interval of time t_a on the aircraft corresponds to the interval of time $n^2 t_a$ on the model. Thus, if the model is half the size of the aircraft so that $n = 0.5$, anything that happens on the aircraft in one second will happen on the model in 0.25 second. The substitution of equations (8), (9), (10), (12), and (13) into either equation (2) or (3) gives

$$H_m = n^3 H_a \tag{14}$$

which states that the heat content of the model is n^3 that of the aircraft. This result is to be expected because the heat capacities of geometrically similar bodies composed of the same material and at the same temperature are proportional to their masses, which in turn are proportional to the cube of their sizes.

Effects of Dissimilar Materials

The foregoing analysis has presumed the structure to be composed of the same material throughout. If different materials are used, then the physical properties of each material in the structure must be included among the physical quantities involved. One material can be selected as a reference material, and use will then be made of dimensionless parameters that express the ratio of the physical properties of the reference material to the corresponding physical properties of each of the other materials. These requirements are automatically fulfilled when each part of the model is made of the same material as the corresponding part of the aircraft structure. The foregoing equations may therefore be considered to apply to structures composed of any number of dissimilar materials.

Summary of Conditions of Thermal Similitude

Under transient heating or cooling, thermal similitude will exist between an aircraft structure and a model thereof when (eqs. (5), (8), (9), and (10))

$$l_m = n l_a$$

$$K_m = K_a$$

$$C_m = C_a$$

$$D_m = D_a$$

The thermal similitude will then be (see eqs. (12), (13), and (14))

$$\theta_m = \theta_a$$

$$t_m = n^2 t_a$$

$$H_m = n^3 H_a$$

Equations (8), (9), and (10) are automatically fulfilled when the model structure is made of the same material as the aircraft structure.

From these similitude equations other useful similitude equations may be derived. Thus, from equations (12) and (5), respectively,

$$\Delta\theta_m = \Delta\theta_a$$

and

$$\Delta l_m = n \, \Delta l_a$$

Dividing the first by the second and taking the limit yields

$$\left(\frac{d\theta}{dl}\right)_m = \frac{1}{n}\left(\frac{d\theta}{dl}\right)_a \tag{15}$$

Thus, the thermal gradient in the model is $1/n$ times that at the corresponding point in the aircraft structure. The similitude of the second derivative of temperature with respect to distance may be obtained as follows:

$$\frac{\Delta\left(\dfrac{d\theta}{dl}\right)_m}{\Delta l_m} = \frac{\dfrac{1}{n}\,\Delta\left(\dfrac{d\theta}{dl}\right)_a}{n \, \Delta l_a}$$

In the limit,

$$\left(\frac{d^2\theta}{dl^2}\right)_m = \frac{1}{n^2}\left(\frac{d^2\theta}{dl^2}\right)_a \tag{16}$$

Thus, the second derivative of temperature with respect to distance in the model is equal to $1/n^2$ times the second derivative of temperature with respect to distance at the corresponding point in the aircraft structure.

II. THERMAL STRESS AND DEFORMATION SIMILITUDE

Formulation of Problem

The objective of part II is to investigate the similarity of thermally induced stress and deformation that can exist between an aircraft structure and a model under transient heating or cooling when the conditions requisite for thermal similitude developed in part I are fulfilled.

Physical Quantities Involved

At the temperature at which an aircraft structure is fabricated, all the parts fit together so that the structure is free of stress when it is not carrying externally applied loads, which include its weight. If the structure is made of a single isotropic material, and its temperature throughout is changed a uniform amount from the assembly temperature, every element of material within the structure will undergo by thermal expansion an identical change in dimensions. The elements will fit together as they did before the temperature change, and therefore no thermally induced stress is developed. Again, if instead of a uniform temperature change, the structure is subjected to a constant temperature gradient throughout, and if the coefficient of thermal expansion is assumed to be constant, each of the elements comprising the structure will undergo a thermal expansion such that all elements fit together, and this result indicates that no thermally induced stress has been produced. On the other hand, if a nonuniform thermal gradient is imposed, the expanded elements do not fit together, so that thermally induced stress is indicated to have been produced. Thus, for the structure composed of the same material throughout, which is the case treated in the analytical theory of thermal stress (ref. 5), it is apparent that thermal stress can develop only when the temperature distribution through the structure is other than constant or linear.

The foregoing argument may be extended to the more general, and more important, case of the structure composed of any number of dissimilar but isotropic materials as follows. If such a structure is subjected to a uniform change of temperature from its assembly temperature, the dissimilar elements will undergo dissimilar thermal expansion so that they no longer fit together; consequently, the development of thermally induced stress is indicated. If the structure is subjected to a constant temperature gradient, the unequal expansion of dissimilar elements will vary with position in the structure. The imposition of a nonuniform thermal gradient produces not only unequal thermal expansion between dissimilar elements but also, as in the case of the structure composed of the same material throughout, expansion of similar elements such that they will no longer fit together.

These considerations show that, in the general case of the structure composed of dissimilar materials, the development of thermally induced stress is in general dependent upon the departure of the temperature from the assembly temperature $\Delta\theta$, the temperature gradient $d\theta/dl$, the second derivative of the temperature with respect to distance $d^2\theta/dl^2$, and for each material involved its coefficient of thermal expansion A and its elastic properties. The elastic properties are the modulus of elasticity in tension or compression E (Young's modulus), the modulus of elasticity in shear G (modulus of rigidity), and Poisson's ratio ν. Only two of these three elastic properties need be considered in that they are theoretically related (ref. 5) by

$$G = \frac{E}{2(1 + \nu)}$$

For simplicity, as well as for the purpose of indicating in subsequent sections that all three of these elastic properties must be identical in corresponding parts of the model and aircraft structures, all three of these elastic properties will be retained.

It is of practical importance to know the closeness with which the stresses approach the elastic limits. Accordingly, the proportional limit in tension or compression σ_p and in shear τ_p are included.

Since it will be shown that the tensile, compressive, and shear stresses in the aircraft and model structures are equal when the similitude requirements are fulfilled, and since these requirements demand that corresponding parts of the model and aircraft structures be composed of identical materials, it may be alternatively considered that the closeness of approach to the elastic limits is implicit in other similitude equations. Again, in order to emphasize that the proportional limits are among the properties defining likeness of material, σ_p and τ_p are explicitly included.

Lastly, among the physical quantities included are the length l, indicative of size and position, the thermally induced linear deformation δ_θ, the thermally induced tensile or compressive stress σ_θ, and the thermally induced shear stress τ_θ.

For brevity, the physical quantities pertaining to each different material in the structure are not written but rather each quantity is understood to pertain to each different material. Accordingly, the physical quantities involved, their units in consistent engineering measure, and their dimensions in terms of length L, mass M, time T, temperature Θ, and heat Q are as follows:

Symbol	Quantity	Units	Dimensions
l	Linear dimension indicative of size	ft	L
δ_θ	Thermal linear deformation	ft	L
σ_θ	Thermal tensile or compressive stress	lb/ft^2	$L^{-1}MT^{-2}$
τ_θ	Thermal shear stress	lb/ft^2	$L^{-1}MT^{-2}$
σ_p	Proportional limit in tension or compression	lb/ft^2	$L^{-1}MT^{-2}$
τ_p	Proportional limit in shear	lb/ft^2	$L^{-1}MT^{-2}$
E	Modulus of elasticity in tension or compression	lb/ft^2	$L^{-1}MT^{-2}$
G	Modulus of elasticity in shear	lb/ft^2	$L^{-1}MT^{-2}$
ν	Poisson's ratio	Dimensionless	Dimensionless
$\Delta\theta$	Temperature change from assembly temperature	°R	Θ
$d\theta/dl$	First derivative of temperature with respect to distance	°R/ft	$L^{-1}\Theta$
$d^2\theta/dl^2$	Second derivative of temperature with respect to distance	°R/ft^2	$L^{-2}\Theta$
A	Coefficient of thermal expansion	1/°R	Θ^{-1}

Dimensional Analysis

By dimensional analysis (refs. 1 and 2), the conditions for thermal-stress and thermal-deformation similitude under transient heating or cooling are found as follows. The above listed physical quantities involved in the phenomenon are observed not to involve the dimension heat Q; therefore, the physical quantities selected as primary quantities need only contain the dimensions length L, mass M, time T, and temperature Θ. Accordingly, the quantities selected as primary quantities are length l, modulus of elasticity in tension or compression E, modulus of elasticity in shear G, and coefficient of thermal expansion A. The resulting π-factors in which the exponents of l, E, G, and A are denoted respectively by a, b, c, and d with numerical subscripts corresponding to the number of the π-factor are then as follows:

$$\delta_\theta \; l^{a_1} \; E^{b_1} \; G^{c_1} \; A^{d_1}$$

$$\sigma_\theta \; l^{a_2} \; E^{b_2} \; G^{c_2} \; A^{d_2}$$

$$\tau_\theta \; l^{a_3} \; E^{b_3} \; G^{c_3} \; A^{d_3}$$

$$\sigma_p \; l^{a_4} \; E^{b_4} \; G^{c_4} \; A^{d_4}$$

$$\tau_p \; l^{a_5} \; E^{b_5} \; G^{c_5} \; A^{d_5}$$

$$\nu \; l^{a_6} \; E^{b_6} \; G^{c_6} \; A^{d_6}$$

$$\Delta\theta \; l^{a_7} \; E^{b_7} \; G^{c_7} \; A^{d_7}$$

$$\frac{d\theta}{dl} \; l^{a_8} \; E^{b_8} \; G^{c_8} \; A^{d_8}$$

$$\frac{d^2\theta}{dl^2} \; l^{a_9} \; E^{b_9} \; G^{c_9} \; A^{d_9}$$

In order to illustrate the determination of the exponents by solution of the conditional equations, the π-factor containing σ_θ is employed as an example. This π-factor is

$$\sigma_\theta \; l^{a_2} \; E^{b_2} \; G^{c_2} \; A^{d_2}$$

Substituting for σ_θ, l, E, G, and A their dimensions in terms of L, M, T, and Θ gives

$$\left(L^{-1}MT^{-2}\right)(L)^{a_2}\left(L^{-1}MT^{-2}\right)^{b_2}\left(L^{-1}MT^{-2}\right)^{c_2}\left(\Theta^{-1}\right)^{d_2}$$

The conditional equations are then:

$$L: \quad -1 + a_2 - b_2 - c_2 = 0$$

$$M: \quad 1 + b_2 + c_2 = 0$$

$$T: \quad -2 - 2b_2 - 2c_2 = 0$$

$$\Theta: \quad -d_2 = 0$$

from whence $a_2 = 0$, $b_2 = -c_2 - 1$, and $d_2 = 0$. The dimensionless parameter resulting from this π-factor is then

$$\frac{\sigma_\theta G^{c_2}}{E^{c_2+1}}$$

where c_2 is any arbitrarily chosen number including zero. Choosing c_2 as zero for simplicity gives

$$\frac{\sigma_\theta}{E}$$

The exponents so determined for the nine π-factors are

$a_1 = -1$	$b_1 = 0$	$c_1 = 0$	$d_1 = 0$
$a_2 = 0$	$b_2 = -1$	$c_2 = 0$	$d_2 = 0$
$a_3 = 0$	$b_3 = -1$	$c_3 = 0$	$d_3 = 0$
$a_4 = 0$	$b_4 = -1$	$c_4 = 0$	$d_4 = 0$
$a_5 = 0$	$b_5 = -1$	$c_5 = 0$	$d_5 = 0$
$a_6 = 0$	$b_6 = 0$	$c_6 = 0$	$d_6 = 0$
$a_7 = 0$	$b_7 = 0$	$c_7 = 0$	$d_7 = 1$
$a_8 = 1$	$b_8 = 0$	$c_8 = 0$	$d_8 = 1$
$a_9 = 2$	$b_9 = 0$	$c_9 = 0$	$d_9 = 1$

The solution yielded by the dimensional analysis is then

$$f\left(\frac{\delta_\theta}{l}, \quad \frac{\sigma_\theta}{E}, \quad \frac{\tau_\theta}{E}, \quad \frac{\sigma_p}{E}, \quad \frac{\tau_p}{E}, \quad \nu, \quad \Delta\theta A, \quad \frac{d\theta}{dl}\,lA, \quad \frac{d^2\theta}{dl^2}\,l^2A\right) = 0 \quad (17)$$

Similitude Equations

With the aircraft structural quantities denoted by the subscript a and the model structural quantities by the subscript m, the similitude equations are immediately written from equation (17) as

$$
\left.
\begin{aligned}
\frac{\delta_{\theta,m}}{l_m} &= \frac{\delta_{\theta,a}}{l_a} \\[6pt]
\frac{\sigma_{\theta,m}}{E_m} &= \frac{\sigma_{\theta,a}}{E_a} \\[6pt]
\frac{\tau_{\theta,m}}{E_m} &= \frac{\tau_{\theta,a}}{E_a} \\[6pt]
\frac{\sigma_{p,m}}{E_m} &= \frac{\sigma_{p,a}}{E_a} \\[6pt]
\frac{\tau_{p,m}}{E_m} &= \frac{\tau_{p,a}}{E_a} \\[6pt]
\nu_m &= \nu_a \\[6pt]
\Delta\theta_m A_m &= \Delta\theta_a A_a \\[6pt]
\left(\frac{d\theta}{dl}\right)_m l_m A_m &= \left(\frac{d\theta}{dl}\right)_a l_a A_a \\[6pt]
\left(\frac{d^2\theta}{dl^2}\right)_m l_m^2 A_m &= \left(\frac{d^2\theta}{dl^2}\right)_a l_a^2 A_a
\end{aligned}
\right\} \qquad (18)
$$

The thermally induced stress and deformation equations (18) are to be satisfied, if possible, simultaneously with the thermal similitude equations (5), (8), (9), (10), (12), (13), (14), (15), and (16). That this is possible is shown by substituting into equations (18) the thermal similitude equation (5) defining scale

$$l_m = nl_a$$

and the following elastic-properties assumptions which are an extension of the thermal similitude assumptions expressed by equations (8), (9), (10), and (11):

$$E_m = E_a \tag{19}$$

$$\nu_m = \nu_a \tag{20}$$

$$\sigma_{p,m} = \sigma_{p,a} \tag{21}$$

$$\tau_{p,m} = \tau_{p,a} \tag{22}$$

$$A_m = A_a \tag{23}$$

Equations (19) to (23) are to be understood as applying to corresponding elements in the model and aircraft. With these substitutions equations (18) reduce to

$$\delta_{\theta,m} = n\delta_{\theta,a} \tag{24}$$

$$\sigma_{\theta,m} = \sigma_{\theta,a} \tag{25}$$

$$\tau_{\theta,m} = \tau_{\theta,a} \tag{26}$$

$$\Delta\theta_m = \Delta\theta_a \tag{27}$$

and

$$\left(\frac{d\theta}{dl}\right)_m = \frac{1}{n}\left(\frac{d\theta}{dl}\right)_a$$

$$\left(\frac{d^2\theta}{dl^2}\right)_m = \frac{1}{n^2}\left(\frac{d^2\theta}{dl^2}\right)_a$$

The latter two equations involving $d\theta/dl$ and $d^2\theta/dl^2$ are identical with the thermal similitude equations (15) and (16). It is to be observed that all higher derivatives of temperature with respect to distance will similarly be compatible. Equation (27) is not only compatible with the thermal similitude equation (12), but is actually a less general form of equation (12). Thus, equations (24), (25), and (26) are the only new equations resulting from the dimensional analysis of the similitude of thermal stresses and deformations. Since they are in no way incompatible

with the thermal similitude equations, it is apparent that both thermal
similitude and thermal stress and deformation similitude are simultaneously
attained in the same model structure when the model is geometrically sim-
ilar to the aircraft structure and every component of the model is made
of identically the same material as the corresponding part of the air-
craft structure.

Equations (25) and (26) show, respectively, that the thermally
induced tension (or compression) stress and shear stress in the model
will be identical to those stresses in the aircraft structure at cor-
responding points. Thus, by appropriate heating tests upon a model
structure, the ability of an aircraft structure to withstand the adverse
effects of aerodynamic heating or cooling can be determined.

Equation (24) relates the linear thermal deformations of the model
and the aircraft. The linear thermal deformations of the model are n
times those of the aircraft structure. Angular deformations in radian
measure are expressed as the ratio of two linear deformations, and hence
the similitude of angular deformations is implicit in equation (24). It
is convenient to have an explicit equation relating the thermally induced
angular deformation of the model $\phi_{\theta,m}$ to that of the aircraft $\phi_{\theta,a}$.
This relation is obtained from equation (24) by writing

$$\frac{\phi_{\theta,m}}{\phi_{\theta,a}} = \frac{\dfrac{\left(\delta_{\theta,m}\right)_1}{\left(\delta_{\theta,m}\right)_2}}{\dfrac{\left(\delta_{\theta,a}\right)_1}{\left(\delta_{\theta,a}\right)_2}} = \frac{\dfrac{n\left(\delta_{\theta,a}\right)_1}{n\left(\delta_{\theta,a}\right)_2}}{\dfrac{\left(\delta_{\theta,a}\right)_1}{\left(\delta_{\theta,a}\right)_2}} = 1$$

from whence

$$\phi_{\theta,m} = \phi_{\theta,a} \tag{28}$$

Thus, the thermally induced angular deformations of the model and air-
craft structures are equal.

Summary of Conditions of Thermal Stress

and Deformation Similitude

When the structural model fulfills the conditions of thermal simil-
itude expressed by equations (5), (8), (9), (10), (12), (13), (14), (15),
and (16), it simultaneously fulfills the requirements for thermally

induced stress and deformation similitude when (see eqs. (19) to (23) and (27))

$$E_m = E_a$$

$$\nu_m = \nu_a$$

$$\sigma_{p,m} = \sigma_{p,a}$$

$$\tau_{p,m} = \tau_{p,a}$$

$$A_m = A_a$$

$$\Delta\theta_m = \Delta\theta_a$$

The similitude of thermally induced stress and deformation will then be (see eqs. (24), (28), (25), and (26))

$$\delta_{\theta,m} = n\delta_{\theta,a}$$

$$\phi_{\theta,m} = \phi_{\theta,a}$$

$$\sigma_{\theta,m} = \sigma_{\theta,a}$$

$$\tau_{\theta,m} = \tau_{\theta,a}$$

III. EXTERNAL LOAD STRESS AND DEFORMATION SIMILITUDE

Formulation of Problem

The objective of part III is to investigate the similitude of stresses and deformations caused by externally applied loads that can exist between an aircraft structure and a model thereof when the model fulfills the requirements for thermal similitude and thermally induced stress similitude developed in parts I and II.

Physical Quantities Involved

An aircraft in curvilinear flight is subject not only to aerodynamic forces and moments, including the thrust, but also to inertial forces and moments arising from the acceleration of gravity and linear and angular accelerations of the aircraft. For simplicity, the linear accelerations due to gravity and motion are customarily added vectorially so that the absolute linear accelerations are obtained. Since an aircraft is a

distended body as distinct from a point mass, the absolute linear accelerations will, in general, vary from point to point within it. Also, its mass is irregularly distributed. Each element of the aircraft structure considered as a free body must be in equilibrium. In accordance with the principle of D'Alembert (ref. 6), the resultants of the externally applied forces and moments on the element are balanced by the inertial forces of the element due to its absolute linear accelerations and its angular accelerations, respectively. Thus, the portions of the externally applied forces and moments effective in producing stress within the element are the applied external forces and moments less the inertial forces and moments. Accordingly, in static tests of aircraft structures simulating flight, the absence of inertial forces and moments due to motion is corrected for by alteration of the externally applied forces representing aerodynamic forces and moments.

If, then, the assumption is made that it is possible in a static test to apply to an aircraft structure external forces and moments so as to generate identically the same stresses and deformations as occur in flight, the problem resolves to a determination of the similitude of stresses and deformations that can exist between such a statically loaded aircraft structure and a model thereof when the model is constructed in accordance with the requirements for thermal similitude and thermal stress similitude developed in parts I and II. Thus viewed, the physical quantities involved, their units in consistent engineering measure, and their dimensions in terms of length L, mass M, and time T are as follows. The dimensions of temperature Θ and heat Q are omitted in that they are not involved. Also, the proportional limit in tension or compression σ_p and the proportional limit in shear τ_p are included among the quantities involved for the same reasons as in the thermal stress and deformation analysis of part II.

Symbol	Quantity	Units	Dimensions
l	Linear dimension indicative of size	ft	L
δ_L	Load linear deformation	ft	L
F	External concentrated load	lb	LMT^{-2}
P	External distributed load	lb/ft^2	$L^{-1}MT^{-2}$
N	External moment	lb-ft	L^2MT^{-2}
σ_L	Load tensile or compressive stress	lb/ft^2	$L^{-1}MT^{-2}$
τ_L	Load shear stress	lb/ft^2	$L^{-1}MT^{-2}$
σ_p	Proportional limit in tension or compression	lb/ft^2	$L^{-1}MT^{-2}$

Symbol	Quantity	Units	Dimensions
τ_p	Proportional limit in shear	lb/ft^2	$L^{-1}MT^{-2}$
E	Modulus of elasticity in tension or compression	lb/ft^2	$L^{-1}MT^{-2}$
G	Modulus of elasticity in shear	lb/ft^2	$L^{-1}MT^{-2}$
ν	Poisson's ratio	Dimensionless	Dimensionless

Dimensional Analysis

By dimensional analyses (refs. 1 and 2), the conditions of similitude of the external loads and moments and the associated structural stresses and deformations are found as follows. Since the physical quantities involve only the dimensions length L, mass M, and time T, the physical quantities selected as primary quantities are length l, modulus of elasticity in tension or compression E, and modulus of elasticity in shear G. The resulting π-factors in which the exponents of l, E, and G are denoted, respectively, by a, b, and c with numerical subscripts corresponding to the number of the π-factor are then as follows:

$$\delta_L \, l^{a_1} \, E^{b_1} \, G^{c_1}$$

$$F \, l^{a_2} \, E^{b_2} \, G^{c_2}$$

$$P \, l^{a_3} \, E^{b_3} \, G^{c_3}$$

$$N \, l^{a_4} \, E^{b_4} \, G^{c_4}$$

$$\sigma_L \, l^{a_5} \, E^{b_5} \, G^{c_5}$$

$$\tau_L \, l^{a_6} \, E^{b_6} \, G^{c_6}$$

$$\sigma_p \, l^{a_7} \, E^{b_7} \, G^{c_7}$$

$$\tau_p \, l^{a_8} \, E^{b_8} \, G^{c_8}$$

$$\nu \, l^{a_9} \, E^{b_9} \, G^{c_9}$$

Determination of the values of the exponents in the same manner as in part II yields:

$a_1 = -1$	$b_1 = 0$	$c_1 = 0$
$a_2 = -2$	$b_2 = -1$	$c_2 = 0$
$a_3 = 0$	$b_3 = -1$	$c_3 = 0$
$a_4 = -3$	$b_4 = -1$	$c_4 = 0$
$a_5 = 0$	$b_5 = -1$	$c_5 = 0$
$a_6 = 0$	$b_6 = -1$	$c_6 = 0$
$a_7 = 0$	$b_7 = -1$	$c_7 = 0$
$a_8 = 0$	$b_8 = -1$	$c_8 = 0$
$a_9 = 0$	$b_9 = 0$	$c_9 = 0$

The solution yielded by the dimensional analysis is then

$$f\left(\frac{\delta_L}{l}, \quad \frac{F}{l^2 E}, \quad \frac{P}{E}, \quad \frac{N}{l^3 E}, \quad \frac{\sigma_L}{E}, \quad \frac{\tau_L}{E}, \quad \frac{\sigma_p}{E}, \quad \frac{\tau_p}{E}, \quad \nu\right) = 0 \qquad (29)$$

Similitude Equations

Denoting, as in preceding sections, the aircraft structure by the subscript a and the model structure by the subscript m, the similitude equations are immediately written from equation (29) as

$$\left.\begin{aligned} \frac{\delta_{L,m}}{l_m} &= \frac{\delta_{L,a}}{l_a} \\[2ex] \frac{F_m}{l_m^2 E_m} &= \frac{F_a}{l_a^2 E_a} \\[2ex] \frac{P_m}{E_m} &= \frac{P_a}{E_a} \end{aligned}\right\} \qquad (30)$$

$$\frac{N_m}{l_m^3 E_m} = \frac{N_a}{l_a^3 E_a}$$

$$\frac{\sigma_{L,m}}{E_m} = \frac{\sigma_{L,a}}{E_a}$$

$$\frac{\tau_{L,m}}{E_m} = \frac{\tau_{L,a}}{E_a}$$

$$\frac{\sigma_{p,m}}{E_m} = \frac{\sigma_{p,a}}{E_a}$$ (30)
Conc.

$$\frac{\tau_{p,m}}{E_m} = \frac{\tau_{p,a}}{E_a}$$

$$\nu_m = \nu_a$$

The external load, moment, stress, and deformation similitude equations (30) are to be satisfied, if possible, simultaneously with the thermal similitude equations (5), (8), (9), (10), (12), (13), (14), (15), and (16) and with the thermal stress and deformation similitude equations (19) to (28). That this is possible and the conditions under which it is possible are found by substituting into equations (30) the thermal similitude equations and the thermal stress and deformation similitude equations, whereupon equations (30) reduce to

$$\delta_{L,m} = n\delta_{L,a}$$ (31)

$$F_m = n^2 F_a$$ (32)

$$P_m = P_a$$ (33)

$$N_m = n^3 N_a$$ (34)

$$\sigma_{L,m} = \sigma_{L,a}$$ (35)

$$\tau_{L,m} = \tau_{L,a} \tag{36}$$

Thus, when the model is geometrically similar to the aircraft structure and constructed of the same materials, the externally applied forces, pressures, and moments (corrected for inertial forces and moments) that must be applied to the aircraft structure in a static test properly to simulate those obtained in flight are related to the externally applied forces, pressures, and moments that must be applied to the model structure by equations (32), (33), and (34), respectively. When this relationship is obtained, the elastic linear deformations of model and aircraft are related by equation (31), the tensile or compressive stresses by equation (35), and the shear stresses by equation (36).

The relation of model and aircraft angular deformations under these externally applied loads may be derived from equation (31) in a manner similar to the derivation of equation (28). Denoting the load induced angular deformation of the model by $\phi_{L,m}$ and that of the aircraft structure by $\phi_{L,a}$,

$$\frac{\phi_{L,m}}{\phi_{L,a}} = \frac{\dfrac{\left(\delta_{L,m}\right)_1}{\left(\delta_{L,m}\right)_2}}{\dfrac{\left(\delta_{L,a}\right)_1}{\left(\delta_{L,a}\right)_2}} = \frac{\dfrac{n\left(\delta_{L,a}\right)_1}{n\left(\delta_{L,a}\right)_2}}{\dfrac{\left(\delta_{L,a}\right)_1}{\left(\delta_{L,a}\right)_2}} = 1$$

and

$$\phi_{L,m} = \phi_{L,a} \tag{37}$$

It is to be noted that the pressure, force, and moment similitude equations (33), (32), and (34) are those associated with geometrically similar bodies by virtue of the fact that the model and aircraft structures fulfill the requirement

$$l_m = n l_a$$

Inasmuch as force is the product of pressure acting over an area,

$$\frac{F_m}{F_a} = \frac{P_m l_m^2}{P_a l_a^2} = \frac{n^2 P_a l_a^2}{P_a l_a^2} = n^2$$

whence

$$F_m = n^2 F_a$$

Since moment is the product of force and distance,

$$\frac{N_m}{N_a} = \frac{F_m l_m}{F_a l_a} = \frac{n^3 F_a l_a}{F_a l_a} = n^3$$

whence

$$N_m = n^3 N_a$$

Summary of External Load and Stress Similitude

When the structural model fulfills the conditions of thermal similitude expressed by equations (5), (8), (9), (10), (12), (13), (14), (15), and (16), and the thermal stress similitude expressed by equations (19) to (28), the model can then simultaneously fulfill similitude with respect to the aircraft structure statically loaded to correctly simulate flight inertial and aerodynamic loading under the conditions (see eqs. (30) to (37))

$$\delta_{L,m} = n\delta_{L,a}$$

$$F_m = n^2 F_a$$

$$P_m = P_a$$

$$N_m = n^3 N_a$$

$$\sigma_{L,m} = \sigma_{L,a}$$

$$\tau_{L,m} = \tau_{L,a}$$

$$\phi_{L,m} = \phi_{L,a}$$

IV. HEATING THE STRUCTURAL MODEL

Formulation of Problem

The problem considered in part IV is that of applying to each element of surface of the structural model the requisite heating and cooling to simulate correctly the aerodynamic heating which the aircraft experiences in flight. The model is assumed to be constructed in accordance with the equations of part I so that similitude with respect to the flow of heat through the structure exists between the model and the aircraft, in accordance with the equations of part II so that there exists similitude between the model and aircraft with respect to the thermally induced stresses and deformations, and in accordance with the equations of part III so that there also exists similitude between the model and aircraft with respect to the stresses and deformations produced by external loads. The thermal character of the model has been fixed by these considerations so that the problem of heating and cooling the model must be compatible with these prior requirements.

These prior requirements with respect to the transfer of heat to or from the model are embodied in two equations. If the surface heat flux q is defined as the quantity of heat per unit of area per unit of time, the relation between the model and aircraft heat fluxes is obtained from equations (5), (13), and (14) by writing

$$q_m = \frac{H_m}{l_m^2 t_m} = \frac{n^3 H_a}{n^2 l_a^2 n^2 t_a} = \frac{1}{n} q_a$$

or

$$q_m = \frac{1}{n} q_a \qquad\qquad (38)$$

Thus, the heat flux into or out of any elemental area of the model surface must be $1/n$ times that of the corresponding element of surface of the aircraft at corresponding instants of time given by equation (13), which is

$$t_m = n^2 t_a$$

In the present state of development of theoretical and experimental boundary-layer information concerning aerodynamic heating, it is not possible to compute reliably the heat flux to each point on the surface of complex bodies like airplanes or guided missiles for a specified trajectory. If the heat flux is determined experimentally for the particular

body traversing a particular trajectory, there remains the problem of
devising apparatus capable of applying to the model in correct time
sequence the proper heat flux which, in general, may vary from positive
to negative, that is, from heating to cooling. It is conceivable that
if the parameters governing the heat flux are properly scaled, the req-
uisite heat flux applied to the model by aerodynamic heating may pos-
sibly be determined by means of a suitable wind tunnel or by free flight
of the model. If such is possible, both the problem of determining the
heat flux and that of applying the heat flux are simultaneously solved;
accordingly, this possibility is investigated in part IV.

Physical Quantities Involved

In the light of present experimental and theoretical knowledge, the
physical quantities involved in the phenomenon of aerodynamic heating of
aircraft in the region of immediate interest, where radiative heat exchange
is negligible compared to forced convective heat exchange, are the fol-
lowing quantities and have the indicated consistent engineering units
and dimensions in terms of length L, mass M, time T, and temperature θ.
The dimension heat Q is not here used because the phenomenon considered
involves the conversion of energy between the mechanical state and the
heat state.

Symbol	Quantity	Units	Dimensions
q	Local heat flux	lb/ft-sec	MT^{-3}
θ_w	Local surface temperature of body	°R	θ
θ_∞	Free-stream temperature	°R	θ
p_∞	Free-stream pressure	lb/ft^2	$L^{-1}MT^{-2}$
ρ_∞	Free-stream mass density	lb-sec^2/ft^4	$L^{-3}M$
V_∞	Free-stream velocity	ft/sec	LT^{-1}
μ_∞	Free-stream absolute viscosity	lb-sec/ft^2	$L^{-1}MT^{-1}$
k_∞	Free-stream thermal conductivity	lb/(sec)(°R)	$LMT^{-3}\theta^{-1}$
c_p	Free-stream specific heat at constant pressure	ft^2/(sec^2)(°R)	$L^2T^{-2}\theta^{-1}$
γ	Ratio of free-stream specific heats	Dimensionless	Dimensionless

Symbol	Quantity	Units	Dimensions
l	Body linear dimension	ft	L
α	Orientation of body relative to stream	radians	Dimensionless
$\dot{\alpha}$	Angular velocity	radians/sec	T^{-1}
$\ddot{\alpha}$	Angular acceleration	radians/sec^2	T^{-2}

Dimensional Analysis

A dimensional analysis may now be performed, the primary physical quantities selected being the velocity V_∞, the gas density ρ_∞, the reference linear dimension l, and the thermal conductivity k_∞. The following π-factors result in which the exponents of V_∞, ρ_∞, l, and k_∞ are denoted respectively by a, b, c, and d with numerical subscripts corresponding to the number of the π-factor:

$$q \, V_\infty{}^{a_1} \, \rho_\infty{}^{b_1} \, l^{c_1} \, k_\infty{}^{d_1}$$

$$\theta_w \, V_\infty{}^{a_2} \, \rho_\infty{}^{b_2} \, l^{c_2} \, k_\infty{}^{d_2}$$

$$\theta_\infty \, V_\infty{}^{a_3} \, \rho_\infty{}^{b_3} \, l^{c_3} \, k_\infty{}^{d_3}$$

$$p_\infty \, V_\infty{}^{a_4} \, \rho_\infty{}^{b_4} \, l^{c_4} \, k_\infty{}^{d_4}$$

$$\mu_\infty \, V_\infty{}^{a_5} \, \rho_\infty{}^{b_5} \, l^{c_5} \, k_\infty{}^{d_5}$$

$$c_p \, V_\infty{}^{a_6} \, \rho_\infty{}^{b_6} \, l^{c_6} \, k_\infty{}^{d_6}$$

$$\gamma \, V_\infty{}^{a_7} \, \rho_\infty{}^{b_7} \, l^{c_7} \, k_\infty{}^{d_7}$$

$$\alpha \, V_\infty{}^{a_8} \, \rho_\infty{}^{b_8} \, l^{c_8} \, k_\infty{}^{d_8}$$

$$\dot{\alpha} \, V_\infty{}^{a_9} \, \rho_\infty{}^{b_9} \, l^{c_9} \, k_\infty{}^{d_9}$$

$$\ddot{\alpha} \, V_\infty{}^{a_{10}} \, \rho_\infty{}^{b_{10}} \, l^{c_{10}} \, k_\infty{}^{d_{10}}$$

Solution of the conditional equations yields the values of the exponents as:

$$a_1 = -3 \qquad b_1 = -1 \qquad c_1 = 0 \qquad d_1 = 0$$

$$a_2 = -3 \qquad b_2 = -1 \qquad c_2 = -1 \qquad d_2 = 1$$

$$a_3 = -3 \qquad b_3 = -1 \qquad c_3 = -1 \qquad d_3 = 1$$

$$a_4 = -2 \qquad b_4 = -1 \qquad c_4 = 0 \qquad d_4 = 0$$

$$a_5 = -1 \qquad b_5 = -1 \qquad c_5 = -1 \qquad d_5 = 0$$

$$a_6 = 1 \qquad b_6 = 1 \qquad c_6 = 1 \qquad d_6 = -1$$

$$a_7 = 0 \qquad b_7 = 0 \qquad c_7 = 0 \qquad d_7 = 0$$

$$a_8 = 0 \qquad b_8 = 0 \qquad c_8 = 0 \qquad d_8 = 0$$

$$a_9 = -1 \qquad b_9 = 0 \qquad c_9 = 1 \qquad d_9 = 0$$

$$a_{10} = -2 \qquad b_{10} = 0 \qquad c_{10} = 2 \qquad d_{10} = 0$$

The solution yielded by the dimensional analysis is then:

$$f\left(\frac{q}{V_\infty^3 \rho_\infty}, \quad \frac{\theta_w k_\infty}{V_\infty^3 \rho_\infty l}, \quad \frac{\theta_\infty k_\infty}{V_\infty^3 \rho_\infty l}, \quad \frac{p_\infty}{V_\infty^2 \rho_\infty}, \quad \frac{\mu_\infty}{V_\infty \rho_\infty l}, \quad \frac{c_p V_\infty \rho_\infty l}{k_\infty}, \quad \gamma, \quad \alpha, \quad \frac{\dot{\alpha} l}{V_\infty}, \quad \frac{\ddot{\alpha} l^2}{V_\infty^2} \right) = 0$$

$$(39)$$

In this derivation of equation (39), the dimensionless parameters have been directly derived in terms of the free-stream parameters, whereas in the analytic boundary-layer theory the parameters governing aerodynamic heating are usually expressed in terms of the local conditions at the point under consideration. It is to be observed that equation (39) may be alternatively derived by dimensional analysis by first finding the dimensionless parameters in terms of the local conditions and then, by a second dimensional analysis, finding their transformation to free-stream conditions.

Identity of Dimensionless Parameters

The dimensionless parameters occurring in equation (39) are not immediately recognizable as corresponding to the dimensionless parameters customarily employed in heat transfer and in aerodynamics. This is because the customary heat-transfer parameters were originally developed from consideration of an incompressible fluid, and the aerodynamic parameters were separately developed without regard to heat transfer. In equation (39) both heat transfer and aerodynamics are combined. The first parameter $q/V_\infty^3 \rho_\infty$ corresponds to the Nusselt number hl/k of heat transfer, where h is the forced convective heat-transfer coefficient, for had h been employed instead of q only the first parameter of equation (39) would be altered, being changed to the Nusselt number. In the second parameter $\theta_w k_\infty/V_\infty^3 \rho_\infty l$ and in the third parameter $\theta_\infty k_\infty/V_\infty^3 \rho_\infty l$, the quantity $V_\infty^3 \rho_\infty l/k_\infty$ has the dimensions of temperature, and thus the second and third parameters together correspond to the thermal ratio θ_w/θ_∞ of the analytic theory of aerodynamic heating (ref. 7). The fourth parameter $p_\infty/V_\infty^2 \rho_\infty$ corresponds to the Mach number V_∞/a_∞, where a_∞ is the velocity of sound in the free stream. Substituting into the fourth parameter the equation for the velocity of sound in a gas, $\dfrac{p_\infty}{\rho_\infty} = \dfrac{a_\infty^2}{\gamma}$, gives $a_\infty^2/\gamma V_\infty^2$. The ratio of the specific heats γ, being a constant, may be dropped from the dimensionless parameter. The remainder is the square of the reciprocal of the Mach number. Since the reciprocal of a dimensionless parameter, or any power or root of the parameter or its reciprocal may be used with equal validity, the fourth parameter corresponds to the Mach number. The fifth parameter $\mu_\infty/V_\infty \rho l$ is the reciprocal of the Reynolds number $V_\infty \rho_\infty l/\mu_\infty$. The sixth parameter $c_p V_\infty \rho_\infty l/k_\infty$ corresponds to the Prandtl number $c_p \mu_\infty/k_\infty$ for $V_\infty \rho_\infty l_\infty$ has the dimensions of μ_∞ as may be seen from the fifth parameter. The seventh parameter, the ratio of the specific heats γ, is a property of the gas employed that is necessitated by the compressibility of the gas and the conversion of energy in the flow between the mechanical and heat states. For brevity, the eighth parameter α is to be understood as representing the several angles defining the orientation relative to the stream, such as the angle of attack and the angle of yaw. In keeping with this abbreviation, the last two parameters, $\dot{\alpha} l/V_\infty$ and $\ddot{\alpha} l^2/V_\infty^2$, are to be understood as representing the several flow angles imposed upon the distended body by its several angular velocities and accelerations. These parameters are thus recognized as being associated with the several stability derivatives.

Similitude Equations

With the aircraft quantities denoted by the subscript a, and the model quantities by the subscript m, the similitude equations may be written from equation (39) as

$$\left(\frac{q}{V_\infty^3 \rho_\infty}\right)_m = \left(\frac{q}{V_\infty^3 \rho_\infty}\right)_a$$

$$\left(\frac{\theta_w k_\infty}{V_\infty^3 \rho_\infty l}\right)_m = \left(\frac{\theta_w k_\infty}{V_\infty^3 \rho_\infty l}\right)_a$$

$$\left(\frac{\theta_\infty k_\infty}{V_\infty^3 \rho_\infty l}\right)_m = \left(\frac{\theta_\infty k_\infty}{V_\infty^3 \rho_\infty l}\right)_a$$

$$\left(\frac{p_\infty}{V_\infty^2 \rho_\infty}\right)_m = \left(\frac{p_\infty}{V_\infty^2 \rho_\infty}\right)_a$$

$$\left(\frac{\mu_\infty}{V_\infty \rho_\infty l}\right)_m = \left(\frac{\mu_\infty}{V_\infty \rho_\infty l}\right)_a$$

$$\left(\frac{c_p V_\infty \rho_\infty l}{k_\infty}\right)_m = \left(\frac{c_p V_\infty \rho_\infty l}{k_\infty}\right)_a$$

$$\gamma_m = \gamma_a$$

$$\alpha_m = \alpha_a$$

$$\left(\frac{\dot{\alpha} l}{V_\infty}\right)_m = \left(\frac{\dot{\alpha} l}{V_\infty}\right)_a$$

$$\left(\frac{\ddot{\alpha} l^2}{V_\infty^2}\right)_m = \left(\frac{\ddot{\alpha} l^2}{V_\infty^2}\right)_a$$

$$(40)$$

Inasmuch as the model is constructed so as to satisfy the thermal similitude equations of part I, the thermal stress and deformation similitude equations of part II, and the equations of similitude of stress and deformation due to external loads of part III, there may be substituted into equations (40) equation (5), which is

$$l_m = n l_a$$

and equation (38), which is

$$q_m = \frac{1}{n} q_a$$

Also, if the model is assumed to be constructed at the same temperature at which the aircraft is constructed, the equation (27)

$$\Delta \theta_m = \Delta \theta_a$$

may be rewritten as

$$\theta_m = \theta_a \qquad\qquad (41)$$

Since equation (41) applies to any point in the structure, it follows that

$$\theta_{w,m} = \theta_{w,a} \qquad\qquad (42)$$

Substituting equations (5), (38), and (42) into equations (40), and then solving equations (40) simultaneously insofar as possible results in

$$k_{\infty,m} = k_{\infty,a}$$

$$\theta_{\infty,m} = \theta_{\infty,a}$$

$$\left(\frac{p_\infty}{V_\infty{}^2 \rho_\infty}\right)_m = \left(\frac{p_\infty}{V_\infty{}^2 \rho_\infty}\right)_a$$

$$\left(\mu_\infty c_p\right)_m = \left(\mu_\infty c_p\right)_a$$

$$\gamma_m = \gamma_a$$

$$\alpha_m = \alpha_a$$

$$\left(\frac{\dot{\alpha}}{V_\infty}\right)_m = \frac{1}{n}\left(\frac{\dot{\alpha}}{V_\infty}\right)_a$$

$$\left(\frac{\ddot{\alpha}}{V_\infty^2}\right)_m = \frac{1}{n^2}\left(\frac{\ddot{\alpha}}{V_\infty^2}\right)_a$$

If, for the moment, the last three of these equations are disregarded, it is apparent by inspection that the remaining five equations can be compatible with the conditions imposed upon the model in parts I, II, and III only if the model is tested in the same gas in which the aircraft flies, namely air, for it is not possible to specify at will the properties of gases. On the assumption, then, that the model is tested in air,

$$c_{p,m} = c_{p,a}$$

and by the general gas equation

$$\left(\frac{p_\infty}{\rho_\infty}\right)_m = R_g\theta_{\infty,m}$$

and

$$\left(\frac{p_\infty}{\rho_\infty}\right)_a = R_g\theta_{\infty,a}$$

where R_g is the gas constant for air. With these relations, together with the first of equations (40) and equation (38), the five equations reduce to

$$k_{\infty,m} = k_{\infty,a} \tag{43}$$

$$\theta_{\infty,m} = \theta_{\infty,a} \tag{44}$$

$$p_{\infty,m} = \frac{1}{n} \, p_{\infty,a} \qquad\qquad (45)$$

$$\rho_{\infty,m} = \frac{1}{n} \, \rho_{\infty,a} \qquad\qquad (46)$$

$$V_{\infty,m} = V_{\infty,a} \qquad\qquad (47)$$

$$\mu_{\infty,m} = \mu_{\infty,a} \qquad\qquad (48)$$

$$c_{p,m} = c_{p,a} \qquad\qquad (49)$$

$$\gamma_m = \gamma_a \qquad\qquad (50)$$

Consider now the three equations that were momentarily disregarded, which, inasmuch as $V_{\infty,m} = V_{\infty,a}$, now reduce to

$$\alpha_m = \alpha_a \qquad\qquad (51)$$

$$\dot{\alpha}_m = \frac{1}{n} \, \dot{\alpha}_a \qquad\qquad (52)$$

$$\ddot{\alpha}_m = \frac{1}{n^2} \, \ddot{\alpha}_a \qquad\qquad (53)$$

The first of these equations, $\alpha_m = \alpha_a$, is in no way incompatible, for it is physically possible at corresponding instants of time, as given by equation (13),

$$t_m = n^2 t_a$$

for the model and the aircraft to have the same attitude; but if $\alpha_m = \alpha_a$ and $t_m = n^2 t_a$, it follows by differentiation that

$$\dot{\alpha}_m = \frac{1}{n^2} \, \dot{\alpha}_a$$

$$\ddot{\alpha}_m = \frac{1}{n^4} \ddot{\alpha}_a$$

Thus, the aerodynamic-heating similitude equation involving α is compatible with the similitude equations of parts I, II, and III, but the aerodynamic-heating similitude equations involving $\dot{\alpha}$ and $\ddot{\alpha}$ are not compatible except when the scale factor n is unity.

The significance of this incompatibility with respect to $\dot{\alpha}$ and $\ddot{\alpha}$ is that, for a model for which $n \neq 1$, it is not possible to simulate by aerodynamic means the effects upon the aerodynamic heating that are attributable to angular velocities and angular accelerations. These effects are alterations of the direction of flow. For example, in regard to angular velocity, a fighter airplane, capable of a maximum rolling velocity of $200°$ per second and having a wing span of 50 feet, when flying in the stratosphere could develop at its wing tip a change in angle of attack of about $2.6°$ at Mach number 2 and about $1.7°$ at Mach number 3. The other angular velocities and associated local flow-direction changes that the airplane could reasonably encounter at speeds at which aerodynamic heating is significant are of much lower magnitude. At the present state of development of theoretical and experimental information about aerodynamic heating, it is not possible to evaluate the importance of such small flow-direction changes, but by virtue of their smallness their effects are probably small. The flow changes associated with angular accelerations are generally of an even smaller magnitude.

Since the aerodynamic method of applying heating and cooling to the structural model is capable of exactly simulating the aerodynamic heating experienced by the aircraft in every respect except those of angular velocities and angular accelerations when $n \neq 1$, which defect is of questionable significance even in the most violent maneuvers, it is apparent that the method is highly desirable.

Summary of Aerodynamic Heating Similitude Equations

A structural model constructed at the same temperature as the aircraft in accordance with the internal heat-flow similitude equations of part I, the thermally induced stress and deformation similitude equations of part II, and the external-load stress and deformation similitude equations of part III can be subjected to aerodynamic heating and cooling exactly simulating those obtained in flight of the aircraft in every respect (except in regard to angular velocities and angular accelerations) when subjected to an airstream under the following conditions (see eqs. (13), (44) to (47), and (51)):

$$t_m = n^2 t_a$$

$$\theta_{\infty,m} = \theta_{\infty,a}$$

$$p_{\infty,m} = \frac{1}{n} \, p_{\infty,a}$$

$$\rho_{\infty,m} = \frac{1}{n} \, \rho_{\infty,a}$$

$$V_{\infty,m} = V_{\infty,a}$$

$$\alpha_m = \alpha_a$$

The following conditions are then automatically satisfied (see eqs. (38), (42), (43), and (48) to (50)):

$$q_m = \frac{1}{n} \, q_a$$

$$\theta_{w,m} = \theta_{w,a}$$

$$k_{\infty,m} = k_{\infty,a}$$

$$\mu_{\infty,m} = \mu_{\infty,a}$$

$$c_{p,m} = c_{p,a}$$

$$\gamma_m = \gamma_a$$

Physical Possibility of Aerodynamic Heating Method

The physical possibility of aerodynamically heating the structural model as distinct from the practical problems involved, which are beyond the scope of this investigation, are now considered.

If the required time-varying air flow is to be supplied by a supersonic wind tunnel, it is foreseeable that the density, pressure, temperature, and velocity in the test section may theoretically be obtained by extensions of the methods currently employed. Such a tunnel will necessarily provide a throat section for which the geometry can be varied during a run. This variable geometry might be accomplished by use of the sliding-block or flexible-wall type of adjustable supersonic wind-tunnel nozzle.

The possibility of employing a structural model in free flight is next considered. At any given time the atmosphere has a variation of density, pressure, and temperature with altitude that is not alterable at will. Equation (44) requires that the free-stream air temperature of the model and aircraft be the same, whereas equations (45) and (46) require that both the free-stream pressure and density of the model of scale n be 1/n of that of the aircraft. These requirements can obviously be met for a model having a value of. n other than unity only in an isothermal atmosphere which is approximated in the stratosphere. If the standard aeronautical atmosphere (ref. 8) and a half-scale model for which n = 0.5 are assumed, flight of the model at the tropopause (altitude, 36,089 geopotential feet) corresponds to flight of the aircraft at an altitude of 50,511 geopotential feet. It is to be recognized, however, that many guided missiles are sufficiently small so that they may be tested as free-flight models at full scale and therefore would not require alteration of altitude from that at which the missile operates.

V. EXTERNAL LOADING OF THE STRUCTURAL MODEL

WHEN AERODYNAMICALLY HEATED

Formulation of the Problem

The aerodynamic method of heating the structural model developed in part IV simultaneously applies aerodynamic loads to the model. A determination of the similitude of these external aerodynamic loads imposed on the model to the aerodynamic loads encountered by the aircraft in flight is therefore necessary in order that properly adjusted external loads may be imposed on the model for correct similitude of the stresses and deformations caused by external loads as discussed in part III.

Physical Quantities Involved

From consideration of both theoretical and experimental information, the physical quantities influencing aerodynamic forces and having the indicated consistent engineering units and dimensions in terms of length L, mass M, and time T are as follows:

Symbol	Quantity	Units	Dimensions
R	Aerodynamic force	lb	LMT^{-2}
l	Linear dimension	ft	L
ρ_∞	Free-stream mass density	lb-sec^2/ft^4	$L^{-3}M$

Symbol	Quantity	Units	Dimensions
V_∞	Free-stream velocity	ft/sec	LT^{-1}
p_∞	Free-stream pressure	lb/ft^2	$L^{-1}MT^{-2}$
μ_∞	Free-stream absolute viscosity	lb-sec/ft^2	$L^{-1}MT^{-1}$
k_∞	Free-stream thermal conductivity	lb/(sec)($^\circ$R)	$LMT^{-3}\theta^{-1}$
c_p	Free-stream specific heat at constant pressure	ft^2/(sec^2)($^\circ$R)	$L^2T^{-2}\theta^{-1}$
γ	Ratio of free-stream specific heats	Dimensionless	Dimensionless
θ_w	Local surface temperature of body	$^\circ$R	θ
θ_∞	Free-stream temperature	$^\circ$R	θ
α	Orientation of body relative to stream	radians	Dimensionless
$\dot\alpha$	Angular velocity relative to stream	radians/sec	T^{-1}
$\ddot\alpha$	Angular acceleration relative to stream	radians/sec^2	T^{-2}

The inclusion of k_∞, c_p, γ, θ_w, and θ_∞ is justified on the grounds that the viscous forces are influenced by the aerodynamic heating. The exclusion of the heat flux q is in accord with equation (39), which shows that q is a function of the other quantities considered. For brevity, as before, α is to be understood as representing the several angles required to define the attitude of the body, and accordingly $\dot\alpha$ and $\ddot\alpha$ are to be understood as representing the first and second derivatives of these angles with respect to time.

Dimensional Analysis

A dimensional analysis may be performed by selecting as the primary physical quantities V_∞, ρ_∞, l, and k_∞. The following π-factors result:

$$R \; V_\infty^{a_1} \; \rho_\infty^{b_1} \; \imath^{c_1} \; k_\infty^{d_1}$$

$$p_\infty \; V_\infty^{a_2} \; \rho_\infty^{b_2} \; \imath^{c_2} \; k_\infty^{d_2}$$

$$\mu_\infty \; V_\infty^{a_3} \; \rho_\infty^{b_3} \; \imath^{c_3} \; k_\infty^{d_3}$$

$$c_p \; V_\infty^{a_4} \; \rho_\infty^{b_4} \; \imath^{c_4} \; k_\infty^{d_4}$$

$$\gamma \; V_\infty^{a_5} \; \rho_\infty^{b_5} \; \imath^{c_5} \; k_\infty^{d_5}$$

$$\theta_w \; V_\infty^{a_6} \; \rho_\infty^{b_6} \; \imath^{c_6} \; k_\infty^{d_6}$$

$$\theta_\infty \; V_\infty^{a_7} \; \rho_\infty^{b_7} \; \imath^{c_7} \; k_\infty^{d_7}$$

$$\alpha \; V_\infty^{a_8} \; \rho_\infty^{b_8} \; \imath^{c_8} \; k_\infty^{d_8}$$

$$\dot{\alpha} \; V_\infty^{a_9} \; \rho_\infty^{b_9} \; \imath^{c_9} \; k_\infty^{d_9}$$

$$\ddot{\alpha} \; V_\infty^{a_{10}} \; \rho_\infty^{b_{10}} \; \imath^{c_{10}} \; k_\infty^{d_{10}}$$

Solution of the conditional equations yields the values of the exponents in the π-factors as

$a_1 = -2$	$b_1 = -1$	$c_1 = -2$	$d_1 = 0$
$a_2 = -2$	$b_2 = -1$	$c_2 = 0$	$d_2 = 0$
$a_3 = -1$	$b_3 = -1$	$c_3 = -1$	$d_3 = 0$
$a_4 = 1$	$b_4 = 1$	$c_4 = 1$	$d_4 = -1$
$a_5 = 0$	$b_5 = 0$	$c_5 = 0$	$d_5 = 0$
$a_6 = -3$	$b_6 = -1$	$c_6 = -1$	$d_6 = 1$
$a_7 = -3$	$b_7 = -1$	$c_7 = -1$	$d_7 = 1$
$a_8 = 0$	$b_8 = 0$	$c_8 = 0$	$d_8 = 0$
$a_9 = -1$	$b_9 = 0$	$c_9 = 1$	$d_9 = 0$
$a_{10} = -2$	$b_{10} = 0$	$c_{10} = 2$	$d_{10} = 0$

The solution yielded by the dimensional analysis is then

$$f\left(\frac{R}{V_\infty^2 \rho_\infty l^2}, \quad \frac{p_\infty}{V_\infty^2 \rho_\infty}, \quad \frac{\mu_\infty}{V_\infty \rho_\infty l}, \quad \frac{c_p V_\infty \rho_\infty l}{k_\infty}, \quad \gamma, \quad \frac{\theta_w k_\infty}{V_\infty^3 \rho_\infty l}, \quad \frac{\theta_\infty k_\infty}{V_\infty^3 \rho_\infty l}, \quad \alpha, \quad \frac{\dot{\alpha} l}{V_\infty}, \quad \frac{\ddot{\alpha} l^2}{V_\infty^2}\right) = 0 \qquad (54)$$

It is observed that, with the exception of the first dimensionless parameter of equation (54), the parameters are the same as those of equation (39) (but for the absence of the $q/V_\infty^3 \rho_\infty$ parameter) and accordingly have similar identities.

The aerodynamic moment J (having the units of pound-feet and the dimensions $L^2 M T^{-2}$) and the aerodynamic pressure p (having the units of pounds per square foot and the dimensions $L^{-1} M T^{-2}$) are dependent upon the same physical quantities as the aerodynamic force R. Therefore, the functional relationship of the dimensionless parameters involving the aerodynamic moment J and that of the aerodynamic pressure p may be obtained by dimensional analyses similar to that for the aerodynamic force R, or, more conveniently, from equation (54). Solving equation (54) for R yields:

$$R = V^2 \rho_\infty l^2 f\left(\frac{p_\infty}{V_\infty^2 \rho_\infty}, \quad \frac{\mu_\infty}{V_\infty \rho_\infty l}, \quad \frac{c_p V_\infty \rho_\infty l}{k_\infty}, \quad \gamma, \quad \frac{\theta_w k_\infty}{V_\infty^3 \rho_\infty l}, \quad \frac{\theta_\infty k_\infty}{V_\infty^3 \rho_\infty l}, \quad \alpha, \quad \frac{\dot{\alpha} l}{V_\infty}, \quad \frac{\ddot{\alpha} l^2}{V_\infty^2}\right)$$

where the functional factor is the aerodynamic force coefficient. Since an aerodynamic moment is the product of a force and a lever arm

$$J = lR$$

whence

$$f\left(\frac{J}{V_\infty^2 \rho_\infty l^3}, \quad \frac{p_\infty}{V_\infty^2 \rho_\infty}, \quad \frac{\mu_\infty}{V_\infty \rho_\infty l}, \quad \frac{c_p V_\infty \rho_\infty l}{k_\infty}, \quad \gamma, \quad \frac{\theta_w k_\infty}{V_\infty^3 \rho_\infty l}, \quad \frac{\theta_\infty k_\infty}{V_\infty^3 \rho_\infty l}, \quad \alpha, \quad \frac{\dot{\alpha} l}{V_\infty}, \quad \frac{\ddot{\alpha} l^2}{V_\infty^2}\right) = 0 \qquad (55)$$

Similarly, an aerodynamic pressure p is

$$p = \frac{R}{l^2}$$

so that

$$f\left(\frac{p}{V_\infty^2\rho_\infty},\quad \frac{p_\infty}{V_\infty^2\rho_\infty},\quad \frac{\mu_\infty}{V_\infty\rho_\infty l},\quad \frac{c_p V_\infty\rho_\infty l}{k_\infty},\quad \gamma,\quad \frac{\theta_w k_\infty}{V_\infty^3\rho_\infty l},\quad \frac{\theta_\infty k_\infty}{V_\infty^3\rho_\infty l},\quad \alpha,\quad \frac{\dot{\alpha}l}{V_\infty},\quad \frac{\ddot{\alpha}l^2}{V_\infty^2}\right)=0 \qquad (56)$$

Similitude Equations

With the model quantities denoted by the subscript m, and the aircraft quantities by the subscript a, the aerodynamic force, moment, and pressure similitude equations are obtained from equations (54), (55), and (56) as:

$$\left.\begin{aligned}
\left(\frac{R}{V_\infty^2\rho_\infty l^2}\right)_m &= \left(\frac{R}{V_\infty^2\rho_\infty l^2}\right)_a \\[2ex]
\left(\frac{J}{V_\infty^2\rho_\infty l^3}\right)_m &= \left(\frac{J}{V_\infty^2\rho_\infty l^3}\right)_a \\[2ex]
\left(\frac{p}{V_\infty^2\rho_\infty}\right)_m &= \left(\frac{p}{V_\infty^2\rho_\infty}\right)_a \\[2ex]
\left(\frac{p_\infty}{V_\infty^2\rho_\infty}\right)_m &= \left(\frac{p_\infty}{V_\infty^2\rho_\infty}\right)_a \\[2ex]
\left(\frac{\mu_\infty}{V_\infty\rho_\infty l}\right)_m &= \left(\frac{\mu_\infty}{V_\infty\rho_\infty l}\right)_a \\[2ex]
\left(\frac{c_p V_\infty\rho_\infty l}{k_\infty}\right)_m &= \left(\frac{c_p V_\infty\rho_\infty l}{k_\infty}\right)_a \\[2ex]
\gamma_m &= \gamma_a \\[2ex]
\left(\frac{\theta_w k_\infty}{V_\infty^3\rho_\infty l}\right)_m &= \left(\frac{\theta_w k_\infty}{V_\infty^3\rho_\infty l}\right)_a
\end{aligned}\right\} \qquad (57)$$

$$\left.\begin{aligned}
\left(\frac{\theta_\infty k_\infty}{V_\infty{}^3 \rho_\infty l}\right)_m &= \left(\frac{\theta_\infty k_\infty}{V_\infty{}^3 \rho_\infty l}\right)_a \\[2em]
\alpha_m &= \alpha_a \\[2em]
\left(\frac{\dot{\alpha} l}{V_\infty}\right)_m &= \left(\frac{\dot{\alpha} l}{V_\infty}\right)_a \\[2em]
\left(\frac{\ddot{\alpha} l^2}{V_\infty{}^2}\right)_m &= \left(\frac{\ddot{\alpha} l^2}{V_\infty{}^2}\right)_a
\end{aligned}\right\} \qquad\qquad (57)\;\text{Conc.}$$

Aerodynamic Force, Moment, and Pressure Relations Under the

Conditions of Aerodynamic-Heating Similitude

Equations (57) express the conditions of similitude that exist between an aircraft and a model with respect to aerodynamic forces, moments, and pressures. It is now desired to determine whether these similitude conditions are compatible with the similitude conditions of part IV that must be fulfilled in order to subject the model to heating by aerodynamic means in simulation of the aerodynamic heating experienced by the aircraft.

The last two similitude equations of equations (57) are observed to involve the angular velocity $\dot{\alpha}$ and the angular acceleration $\ddot{\alpha}$. As discussed in part IV, the aerodynamic method of heating the model is not capable of satisfying the heating similitude with respect to $\dot{\alpha}$ and $\ddot{\alpha}$ and accordingly provides heating simulation only when the portion of the heating attributable to $\dot{\alpha}$ and $\ddot{\alpha}$ is negligible relative to the remainder of the heating. Consequently, $\dot{\alpha}$ and $\ddot{\alpha}$ are assumed to be zero, whereupon the last two similitude equations of equations (57) vanish.

It is next observed that of equations (57) the fourth through the tenth are identical to the second through the eighth of the aerodynamic-heating similitude equations (40). Therefore, with respect to these similitude equations, the imposing upon the model of heating by the aerodynamic means of part IV simultaneously satisfies these similitude conditions with respect to the attendant aerodynamic forces, moments, and pressures imposed on the model. Accordingly, there remains only the first three of equations (57) to be considered, which are

$$\left(\frac{R}{V_\infty{}^2\rho_\infty l^2}\right)_m = \left(\frac{R}{V_\infty{}^2\rho_\infty l^2}\right)_a \tag{58}$$

$$\left(\frac{J}{V_\infty{}^2\rho_\infty l^3}\right)_m = \left(\frac{J}{V_\infty{}^2\rho_\infty l^3}\right)_a \tag{59}$$

$$\left(\frac{p}{V_\infty{}^2\rho_\infty}\right)_m = \left(\frac{p}{V_\infty{}^2\rho_\infty}\right)_a \tag{60}$$

By the substitution of equations (5), (46), and (47), equations (58), (59), and (60) reduce to

$$R_m = nR_a \tag{61}$$

$$J_m = n^2 J_a \tag{62}$$

$$p_m = \frac{1}{n} p_a \tag{63}$$

When the model is subjected to heating by the aerodynamic means of part IV, the forces, moments, and pressures experienced by the model are given, respectively, by equations (61), (62), and (63). These forces, moments, and pressures are to be compared with the forces, moments, and pressures found in part III to be required for simulation of internal stresses due to externally applied aerodynamic loads.

For simulation of internal stresses due to externally applied aerodynamic forces, the required force similitude is given by equation (32) of part III as

$$F_m = n^2 F_a$$

From this equation and equation (61), since $R_a = F_a$, it is found by substitution that

$$R_m = \frac{1}{n} F_m \tag{64}$$

Thus, when the model is heated by the aerodynamic means described in part IV, the aerodynamic forces imposed upon the model are $1/n$ times as great as those required for simulation of internal stresses due to externally applied aerodynamic loads.

For simulation of internal stresses due to externally applied aerodynamic moments the required moment similitude is given by equation (34) of part III as

$$N_m = n^3 N_a$$

If this equation is compared with equation (62), since $J_a = N_a$, it is found by substitution that:

$$J_m = \frac{1}{n} N_m \qquad (65)$$

Thus, when the model is heated by the aerodynamic means discussed in part IV, the aerodynamic moments imposed upon the model are $1/n$ times as great as those required for simulation of internal stresses due to externally applied aerodynamic moments.

For simulation of internal stresses due to externally applied aerodynamic pressures the required pressure similitude is given by equation (33) of part III as

$$P_m = P_a$$

Comparison of this equation with equation (63) shows that, since $p_a = P_a$,

$$p_m = \frac{1}{n} P_m \qquad (66)$$

Therefore, when the model is heated by the aerodynamic means described in part IV, the aerodynamic pressures imposed upon the model are $1/n$ times as great as those required for simulation of internal stresses due to externally applied aerodynamic pressures.

Inspection of equations (64), (65), and (66) shows that, when $n < 1$, the forces, moments, and pressures imposed on the model will exceed those required for stress due to external aerodynamic loads, even when the corrections for inertial effects and gravity are disregarded. It is apparent, then, that the model subjected to heating in a wind tunnel will require that corrective loads be applied to it by some mechanical means. It

follows that the possibility of testing the model for which $n < 1$ in free flight, as discussed in part IV, is precluded by inability to correct the external loads to those required.

Consider next the case where $n = 1$. This degenerate case is in essence the testing of the aircraft structure itself, and hence in the wind tunnel similitude to flight exists except for inertial loads and the effects of angular velocity and angular acceleration.

Langley Aeronautical Laboratory,
      National Advisory Committee for Aeronautics,
            Langley Field, Va., July 15, 1957.

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